

# Synchronization and Inertial Frames

C. VIAZMINSKY

INTERNATIONAL INSTITUTE OF THEORETICAL AND APPLIED PHYSICS,  
IOWA STATE UNIVERSITY, AMES, IA 50011, USA  
AND DEPARTMENT OF PHYSICS, UNIVERSITY OF ALEPPO, SYRIA

February 2, 2008

**ABSTRACT.** In classical mechanics a procedure for simultaneous synchronization in all inertial frames is consistent with the Galilean transformation. However, if one attempts to achieve such synchronization utilizing light signals, then he will be facing on one hand a break down of absolute simultaneity, and on the other hand, a self-contradictory transformation that has Lorentz transformation, or the confinement of Lorentz transformation to the velocity of light, as the only possible ways that resolve the contradiction. The current work constitutes a smooth transition from traditional to relativistic vision of mechanics, and therefore is quite appealing from pedagogical point of view.

## 1. INTRODUCTION

Time in Newtonian mechanics is considered as an absolute entity, so that one can choose an instant  $t_0$  which can be taken as the initial time in all inertial frames, and such that when an event occurs at an instant  $t$  in one inertial frame then it occurs also at the same instant  $t$  in all inertial frames. The inquiry of how, in principle, can one synchronize the clocks in all inertial frames receives a number of plausible answers. One possible hypothetical procedure is as follows: it is sufficient to synchronize clocks in one inertial frame, say  $S$ , with a master clock at some point in  $S$ , say the origin  $O$ , using a stream of particles emitted radially from  $O$  with the same velocity  $C$ , so that on first receiving the stream, an observer at a point  $P$  sets his clock on  $t = t_0 + |\vec{r}|/C$ , where  $\vec{r}$  is the position vector of the point  $P$  and  $t_0$  is the time read by the master clock when the stream starts[7]. In this way the space is furnished with a set of synchronized clocks which can be adopted in all inertial frames. In fact it is straight forward to see, using the Galilean transformation for the coordinates and the velocity, that the procedure we have just described does synchronize also the clocks in every other inertial frame with the master clock at the origin of  $S$ , and accordingly with each other. However if it turns out that the law of velocity addition in Galilean transformation is not applicable to the stream of particles then synchronization is beyond reach because the concept of absolute simultaneity ceases to be valid.

We present here a procedure for synchronization using a stream of photons, and show that in spite of one's attempt to adhere to the familiar concepts of Newton

mechanics, the shortcomings of these concepts are revealed when synchronization is attempted for more than one inertial frame. We shall adopt from start the experimental fact [3] that the velocity of light is independent of the state of relative motion between the source and the observer, which is equivalent to say that this velocity is a constant  $C$ , which is the same in all inertial frames.

## 2. SYNCHRONIZATION BY LIGHT SIGNALS

Let  $S$  and  $S'$  be two inertial frames comprising two systems of rectangular Cartesian coordinates  $oxyz$  and  $o'x'y'z'$  respectively, in standard configuration. Assume that  $S'$  is moving relative to  $S$  in the direction of the  $x$ -axis with a uniform velocity  $u$  ( $u > 0$ ). When the plane  $x' = 0$  in  $S'$  coincide with the plane  $x = 0$  in  $S$ , every observer in the plane  $x' = 0$  sets his clock  $T' = 0$  and simultaneously emits a pulse of light of short duration parallel to the  $x'$ -axis. The assembly of these pulses generates a plane wave propagating in the  $\pm x'$  directions with velocity  $C$ . The furthest wave front appears to the observer  $O'$  at an instant  $T'$  as occupying two planes  $|x'| = CT'$ , and hence all observers in these two planes have to set their clocks at  $T'$ . Since time flows uniformly in  $S'$ , all observers in the region  $\{X' : |X'| \leq CT'\}$  have the same clock reading, namely  $T'$ .

For the  $S$ -observer the plane wave we have just prescribed appears as originating from the plane  $x = 0$  and travelling in the  $\pm x$ -directions with velocity  $C$ , for the light velocity has been assumed to be constant and independent of the source's motion. The observer  $O$  uses the Galilean transformation to deduce that the equation of the light's front is

$$x = x' + uT', \quad (1)$$

which represents two planes corresponding to  $x' = \pm CT'$  in  $S'$ .

An  $S$ -observer at a location  $(x, y, z)$ , on first receiving the wave has to set his clock at  $T = |x|/C$ . Dividing (1) by  $C$  and making use of  $T' = |x'|/C$  we get the equation

$$T = T' + ux'/C^2, \quad (2)$$

which relates the initial time settings in both frames for the points lying on the wave front. Although at an instant  $T'$  in  $S'$ , all clocks in the region  $|X'| \leq CT' = |x'|$  record the same time  $T'$ , the corresponding clocks in  $S$  in the region  $-x' + uT' \leq X \leq |x'| + uT'$ , or equivalently  $(-C + u)T' \leq X \leq (C + u)T'$ , do not read the same time. In fact all those corresponding to  $x' = CT'$  read  $T = T'(1 + u/C)$ , whereas those corresponding to  $x' = -CT'$  read  $T = T'(1 - u/C)$ . In other words time is no more absolute in  $S$ , i.e. in the system which performed synchronization using the Galilean transformation, and at the same time accepting that the velocity of light waves emitted from the plane  $x' = 0$  in  $S'$  is also equal to  $C$  in  $S$ . The peculiar result

concerning time in  $S$  is expected, because the velocity in classical mechanics cannot be independent of the inertial frame relative to which it is measured.

### 3. LORENTZ TRANSFORMATION

In matrix notation we write the transformation given by (1,2), together with the obviously valid relations  $y = y', z = z'$  as

$$(Txyz)^\sim = A(T'x'y'z')^\sim \quad (3)$$

where the symbol  $(\cdot)^\sim$  denoted the transpose of a four-vector, and  $A$  is the transformation matrix

$$A = \begin{bmatrix} 1 & u/C^2 & 0 & 0 \\ u & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

On physical grounds, and on interchanging the roles of the  $S$  and  $S'$ -observers, we find that the transformation from  $S$  to  $S'$  :

$$(T'x'y'z')^\sim = A(Txyz)^\sim, \quad (5)$$

results through replacing  $u$  in  $A$  by  $(-u)$  to obtain

$$A' = \begin{bmatrix} 1 & -u/C^2 & 0 & 0 \\ -u & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6)$$

The composite transformation (3) and (5), i.e. from  $S'$  to  $S$  and back to  $S'$  must certainly be the identity transformation. We have however

$$AA' = \begin{bmatrix} (1 - u^2/C^2)I_2 & 0 \\ 0 & I_2 \end{bmatrix}, \quad (7)$$

which is a contradiction. The way out of this contradiction is to scale the right hand-sides of the expressions of  $T$  and  $x$  as given by (1) and (2), and also the similar expressions of  $T'$  and  $x'$ , through multiplication by  $\gamma = (1 - u^2/C^2)^{-1/2}$ , to obtain

$$\begin{aligned} x &= \frac{x' + uT'}{\sqrt{1 - u^2/C^2}}, \quad y = y', \quad z = z' \\ T &= \frac{T' + ux'}{\sqrt{1 - u^2/C^2}}. \end{aligned} \quad (8)$$

Lorentz transformation is obtained on postulating that the last relations are valid for an arbitrary moment  $t'$  in  $S'$ , i.e. we have to replace  $T'$  in (8) by an arbitrary  $t'$  and the corresponding  $T$  by  $t$ . For more details concerning Lorentz transformation and their properties we refer to [1, 4, 5, 6]. As far as our work is concerned it is important to note that Lorentz transformation is a point transformation in a four dimensional manifold  $(t, x, y, z)$ , in the sense that each point (event)  $(t, x, y, z)$  in  $S$  is described by a unique point  $(t', x', y', z')$  in  $S'$ , and hence the classical vision of absolute time is no more valid. Therefore our goal of achieving simultaneous synchronization of inertial frames is proved impossible, and instead, every frame is synchronized independently. For more details about the new aspects of space and time we refer to reference[2].

#### 4. AN ALTERNATIVE RESULT

For the points of the synchronization wave front, the relations

$$|x| = CT, \quad |x'| = CT' \quad (9)$$

are valid. Hence we may write (1) and (2), according to  $x'$  being positive or negative in either form

(i) For  $x' > 0$

$$x = (1 + u/C)x', \quad T = (1 + u/C)T'. \quad (10)$$

(ii) For  $x' < 0$

$$x = (1 - u/C)x', \quad T = (1 - u/C)T'. \quad (11)$$

Since  $u/C < 1$ , it is evident that  $x$  and  $x'$  are both positive or both negative. An identical statement is also valid for  $T$  and  $T'$ . It is evident, in both cases (i) and (ii), that the two equations obtained are linearly dependent, and therefore are reducible to one equation. In fact the second equation in each case is obtained through dividing both sides of the first equation by  $C$  or  $(-C)$ . We shall therefore confine our attention to the second equation in each case.

Due to symmetry the transformation from  $S$  to  $S'$  is obtained by replacing  $u$  by  $(-u)$  in (10) and (11); and hence must be

(i) For  $x > 0$

$$T' = (1 - u/C)T \quad (12)$$

(ii) For  $x < 0$

$$T' = (1 + u/C)T \quad (13)$$

But these are not the inverse transformation as deduced from (10) and (11). In fact by (10-13) we have the contradiction:

$$\begin{aligned} \text{For } x > 0 \quad T &= (1 + u/C)T' = (1 - u^2/C^2)T \\ \text{For } x < 0 \quad T &= (1 - u/C)T' = (1 - u^2/C^2)T, \end{aligned}$$

which is resolved as before by multiplying the right hand-sides of the relations (10-13) by the factor  $\gamma = (1 - u^2/C^2)^{-1/2}$  to obtain

$$x = ax', \quad T = aT' \quad (x' > 0), \quad (14)$$

$$x = x'/a, \quad T = T'/a \quad (x' < 0), \quad (15)$$

where

$$a = \sqrt{(C + u)/(C - u)}. \quad (16)$$

Replacing  $u$  by  $(-u)$  gives the inverse transformation

$$x' = x/a, \quad T' = T/a \quad (x > 0), \quad (17)$$

$$x' = ax, \quad T' = aT \quad (x < 0). \quad (18)$$

The transformation (14,15) or the inverse transformation (17,18) gives the relations between the coordinates of an arbitrary point in both frames the moment it receives the wave of synchronization, or equivalently, the relation between the time settings by the  $S$  and  $S'$ -observers at that point when they first receive the wave.

It is important to notice that the transformation we have just found may be deduced from Lorentz transformation. Actually if we set  $|x'| = CT'$  in (8) we get the transformation (14,15). Note that for  $|x'| = CT'$  which is the equation of the fronts of two waves originating simultaneously from the plane  $x' = 0$  and travelling in opposite directions, are transformed by Lorentz transformation (8) into two planes given by (14,15). The last two planes are not symmetric with respect to  $x = 0$ , because the simultaneous events  $(T', x = \pm CT', y', z')$  in  $S'$  are not simultaneous in  $S$ .

Perhaps it is worthy to mention that the relations (14,15) are equivalent to the special Lorentz transformation (8) in which we still have the constraints  $|x'| = CT'$ , and which is applicable only to the positions occupied by the light front and the corresponding times. Lorentz transformation results indeed from (8) only when we abandon this constraint and deal with  $x'$  and  $T'$  as independent variables.

Our approach therefore, leads either to Lorentz transformation itself, or to Lorentz transformation applicable only to the light propagation. Only a daring step in which the coordinates and time are liberated from the constraint, we have mentioned, leads to Lorentz transformation.

## 5. CONCLUSION

The Galilean transformation is completely compatible with the concept of absolute time in classical mechanics. However, when one attempts to incorporate the source-independent constant velocity of light in Galilean transformation then he will be faced from start by a type of transformation, namely the wrong intermediate transformation (1,2) which terminates the concept of simultaneity and accordingly absolute time.

Following up this result and accepting the symmetric roles of inertial frames we deduce that the transformation from  $S$  to  $S'$  result from (1,2) by simply replacing  $u$  by  $(-u)$ . The final transformation we are seeking must be satisfied by the coordinates of any event that are measured by observers in the two frames. The mathematical expression of the last statement is that the transformation from one frame to another must be inverse to each other. This requirement determines a factor ( $\gamma$ ) by which we have to scale the right hand-sides of the intermediate transformation to obtain formally the Lorentz transformation. Although, it is known to every physicist that Lorentz transformation is the correct substitute of the Galilean transformation, the method we have followed to derive this transformation is new and has nice features manifested in the smooth transition from Newtonian to relativistic concepts.

## REFERENCES

- [1] M. Born, Einstein's Theory of Relativity (Dover, New York, 1962).
- [2] P.C.W. Davies, Space and Time in Modern Universe (Cambridge University Press, 1984).
- [3] T.A. Fillipas and J.C. Fox, Physical Review B135(1964) 1071.
- [4] A.P. French, Special Relativity (Butler and Turner LTD., Frome and London, 1968).
- [5] L. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, New York, 1980).
- [6] F.D. Lawden, Tensor Calculus and Relativity (Chapman and Hall, London, 1975).
- [7] W. Rindler, Essential Relativity (Springer-Verlag, New York, 1979).